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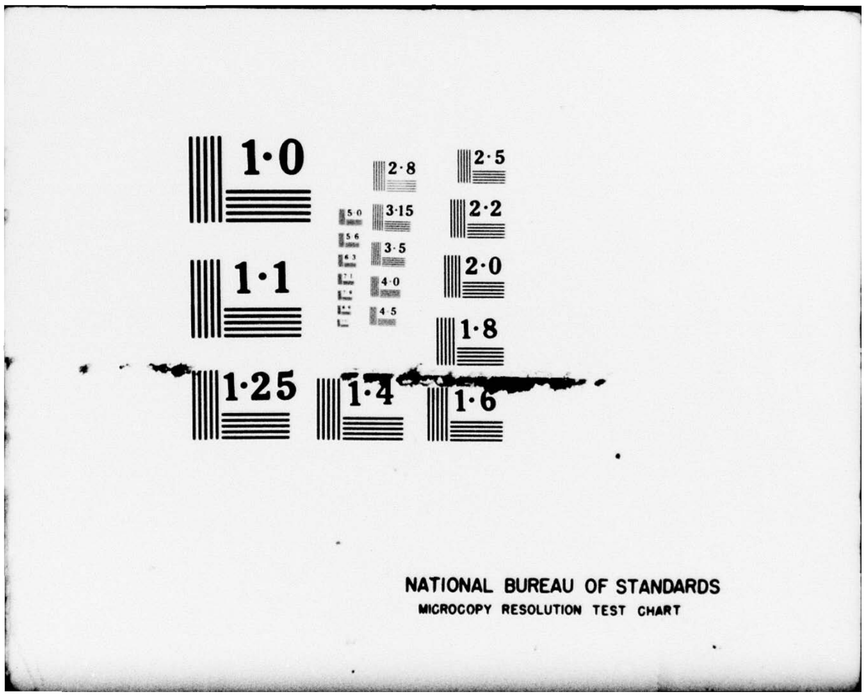
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The Second Moments of the Absorption Times in the Markov Renewal Branching Process

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ABSTRACT

We derive explicit formulas for the second moments of the absorption time matrices in the Markov Renewal Branching process. These formulas may easily be computationally implemented and are useful in the iterative computation of the semi-Markov matrices, which give the distributions of the duration and of the number of customers served in a busy period of a great variety of complex queueing models.

KEY WORDS

Markov renewal branching process, queues, semi-Markov service times, computational probability, busy period

I. Introduction

This note is a sequel to [1], where we discussed the moments of semi-Markov matrices associated with the time till extinction in a Markov renewal branching process. We refer to [1] for the motivation and a general description of the problem under discussion. In queueing theory, the semi-Markov matrices of interest are related to the number of customers served and to the time duration of the busy period. See e.g. [2].

Recalling only the most essential definitions from [1], we consider a sequence $\{\hat{A}_n(\cdot)\}$ of substochastic semi-Markov matrices, with Laplace-Stieltjes transforms $\{A_n^*(s)\}$, such that the sum

$$(1) \quad \hat{A}(x) = \sum_{n=0}^{\infty} \hat{A}_n(x), \qquad x \geq 0,$$

is an irreducible, regular stochastic semi-Markov matrix. The invariant probability vector of $A=\hat{A}(+\infty)$ is denoted by $\underline{\pi}$ and the vector $\underline{\beta}$ is defined by

$$(2) \quad \underline{\beta} = \sum_{n=1}^{\infty} nA_n\underline{e},$$

where $A_n=\hat{A}_n(+\infty)$, for $n \geq 0$, and $\underline{e}=(1,1,\ldots,1)'$.

Throughout this paper, we shall assume that the irreducibility assumptions, stated in [1], hold and also that

$$(3) \quad \rho = \underline{\pi}\underline{\beta}<1.$$

There then exists a unique sequence of substochastic semi-Markov matrices $\{\hat{G}_n(\cdot)\}$, such that the transform matrix

$$(4)\quad G^*(z,s) = \sum_{n=1}^{\infty} z^n \int_0^{\infty} e^{-sx}\, d\hat{G}_n(x),$$

defined for $0 \le z \le 1$, $s \ge 0$, satisfies the nonlinear matrix equation

$$(5)\quad G^*(z,s) = z \sum_{n=0}^{\infty} A_n^*(s)[G^*(z,s)]^n.$$

Moreover the matrix $G = G^*(1,0) = \sum_{n=1}^{\infty} \hat{G}_n(+\infty)$, is a positive stochastic matrix, which satisfies

$$(6)\quad G = \sum_{n=0}^{\infty} A_n G^n.$$

The invariant probability vector of G is denoted by $\underline{g}$ and the matrix $\tilde{G}$ is defined by $\tilde{G}_{ij} = g_j$, for $1 \le i,j \le m$. The matrix $I - G + \tilde{G}$ is known to be nonsingular. The matrices M and N are defined by

$$(7)\quad M = \int_0^{\infty} x\, d\sum_{n=1}^{\infty} \hat{G}_n(x) = -\left[\frac{d}{ds} G^*(1,s)\right]_{s=0},$$

$$N = \sum_{n=1}^{\infty} n\hat{G}_n(+\infty) = \left[\frac{d}{dz} G^*(z,0)\right]_{z=1},$$

and $\underline{\mu} = M\underline{e}$, $\underline{\nu} = N\underline{e}$.

For future reference, we define

$$(8)\quad B(1) = \sum_{n=1}^{\infty} nA_n, \qquad \underline{\beta} = B(1)\underline{e},$$

$$B(2) = \sum_{n=2}^{\infty} n(n-1)A_n, \qquad \underline{\beta}_2 = B(2)\underline{e},$$

$$C(1) = \int_0^{\infty} x\, d\hat{A}(x) = -\left[\frac{d}{ds} A^*(s)\right]_{s=0}, \qquad \underline{\beta}^* = C(1)\underline{e},$$

$$C(2) = \int_0^{\infty} x^2\, d\hat{A}(x) = \left[\frac{d^2}{ds^2} A^*(s)\right]_{s=0}, \qquad \underline{\beta}_2^* = C(2)\underline{e},$$

$$D = \sum_{n=1}^{\infty} n\int_0^{\infty} x\, d\hat{A}_n(x) = -\sum_{n=1}^{\infty} nA_n^{*\prime}(0+), \qquad \underline{\delta} = D\underline{e},$$

$$E(1) = -\sum_{n=0}^{\infty} A_n^{*\prime}(0+)G^n,$$

$$E(2) = \sum_{n=0}^{\infty} A_n^{*\prime\prime}(0+)G^n.$$

These matrices will be assumed to be known and finite. We note that if G is known accurately, then E(1) and E(2) can in principle be computed, with $E(1)\underline{e}=\underline{\beta}^*$ and $E(2)\underline{e}=\underline{\beta}_2^*$, serving as accuracy checks.

By routine differentiations in Equation (5) and using results in [1], the matrices M and N are the unique solutions, respectively to the matrix equations

$$\text{(9)} \qquad M = E(1)+\sum_{n=1}^{\infty} A_n \sum_{\nu=0}^{n-1} G^{\nu} M G^{n-1-\nu},$$

$$N = G+\sum_{n=1}^{\infty} A_n \sum_{\nu=0}^{n-1} G^{\nu} N G^{n-1-\nu},$$

and the vectors $\underline{\mu}$ and $\underline{\nu}$ are given explicitly by

$$\text{(10)} \qquad \underline{\mu} = (I-G+\tilde{G})[I-A+\tilde{G}-\Delta(\underline{\beta})\tilde{G}]^{-1}\underline{\beta}^*,$$

$$\underline{\nu} = (I-G+\tilde{G})[I-A+\tilde{G}-\Delta(\underline{\beta})\tilde{G}]^{-1}\underline{e},$$

where $\Delta(\underline{\beta})=\mathrm{diag}(\beta_1,\dots,\beta_m)$.

Powerful accuracy checks on numerical computations are provided by the formulas

$$\text{(11)} \qquad \underline{g}\underline{\mu} = (1-\rho)^{-1}\underline{\pi}\underline{\beta}^*, \qquad \underline{g}\underline{\nu} = (1-\rho)^{-1}.$$

The purpose of this paper is as follows. In discussing the steady-state probability distributions of a variety of queueing models, it turns out that only the vectors $\underline{g}$, $\underline{\mu}$ and $\underline{\nu}$ and the matrix G are of essential computational importance. Even for matrices A_n of fairly high order, the amount of computation is small to moderate.

In contrast, the numerical computation of the semi-Markov matrix $\hat{G}(x) = \sum_{n=1}^{\infty} \hat{G}_n(x)$, for $x \geq 0$, and of the sequence $\{\hat{G}_n(+\infty)\}$ is a major task. The most promising algorithms involve iterations of the nonlinear operator, corresponding to the inverted form of Equation (5).

Such algorithms require however that a priori cut-off points on the tails of the probability distributions be computed. With information on the second moments of $\hat{G}(\cdot)$ and $\{\hat{G}_n(\infty)\}$ available, we can limit computation to terms up to three or four standard deviations beyond the mean. We therefore proceed to a discussion of the matrices M_2 and N_2, defined by

$$(12)\quad M_2 = \int_0^{\infty} x^2 d\hat{G}(x) = \left[\frac{d^2}{ds^2} G^*(1,s)\right]_{s=0},$$

$$N_2 = \sum_{n=2}^{\infty} n(n-1)\hat{G}_n(+\infty) = \left[\frac{d^2}{dz^2} G^*(z,0)\right]_{z=1},$$

and the vectors $\underline{\mu}_2 = M_2\underline{e}$, and $\underline{\nu}_2 = N_2\underline{e}$.

Although the derivation of explicit formulas for the latter is primarily a matter of calculation, the intermediate steps are sufficiently involved that it appears worthwhile to have the resulting expressions available in the literature.

II. The Matrices M_2 and N_2.

By routine differentiations in Formula (5), we obtain the equations

$$(13)\quad M_2 - \sum_{n=1}^{\infty} A_n \sum_{\nu=0}^{n-1} G^{\nu} M_2 G^{n-1-\nu} =$$

$$E(2) - 2\sum_{n=1}^{\infty} A_n^{*\prime}(0+) \sum_{\nu=0}^{n-1} G^{\nu} M G^{n-1-\nu}$$

$$+2\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2} \sum_{r=0}^{\nu} G^{r} M G^{\nu-r} M G^{n-2-\nu},$$

and

$$(14)\quad N_2 - \sum_{n=1}^{\infty} A_n \sum_{\nu=0}^{n-1} G^{\nu} N_2 G^{n-1-\nu} =$$

$$2N - 2G + 2\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2} \sum_{r=0}^{\nu} G^{r} N G^{\nu-r} N G^{n-2-\nu}.$$

For practical purposes, the explicit expressions, given below, for $\underline{\mu}_2$ and $\underline{\nu}_2$ will be sufficient. The numerical solution of Equations (13) and (14) is feasible, whenever the right-hand sides can be computed accurately. The following results convey useful information on such computation.

Lemma 1

For any square matrix X, we have

$$(15)\quad \lim_{n\to\infty} n^{-2} \sum_{\nu=0}^{n-2} \sum_{r=0}^{\nu} G^{r} X G^{\nu-r} X G^{n-2-\nu} = \tilde{G}X\tilde{G}X\tilde{G} = (\underline{g}X\underline{e})^2 \tilde{G}.$$

Proof

Writing $U_n = \sum_{\nu=0}^{n-1} G^{\nu} X G^{n-1-\nu}$, and noting that by interchanging the order of the summations, we obtain

$$(16)\quad T_n = \sum_{\nu=0}^{n-2} \sum_{r=0}^{\nu} G^{r} X G^{\nu-r} X G^{n-2-\nu} = \sum_{\nu=0}^{n-2} G^{\nu} X U_{n-\nu-1},$$

so that

$$(17)\quad n^{-2} T_n = n^{-1} \sum_{\nu=0}^{n-2} \frac{n-\nu}{n} G^{\nu} X (n-\nu)^{-1} U_{n-\nu-1}.$$

In [1], Thm 4, we proved that $n^{-1}U_n \to (\underline{g}X\underline{e})\tilde{G}$. Using this result and repeating verbatim the same proof for the expression in (17), the stated result follows.

Lemma 1 shows that the truncation error at the index n=K in the infinite sum $\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2} \sum_{r=0}^{\nu} G^r M G^{\nu-r} M G^{n-2-\nu}$, is approximately equal to

$$(18) \quad (1-\rho)^{-2}(\underline{\pi}\underline{\beta}^*)^2 \sum_{n=K+1}^{\infty} n^2 A_n.$$

Similar error estimates apply to the other infinite series.

The matrices U_n and T_n can be computed recursively for use in the computation of the right hand sides. We have the recurrence relations:

$$(19) \quad U_1 = X, \qquad U_n = XG^{n-1} + GU_{n-1}, \qquad \text{for } n \geq 2,$$

$$T_2 = X^2, \qquad T_n = XU_{n-2} + GT_{n-1}, \qquad \text{for } n \geq 3.$$

III. The Vectors $\underline{\mu}_2$ and $\underline{\nu}_2$.

Theorem 1

The vectors $\underline{\mu}_2$ and $\underline{\nu}_2$ are given by

$$(20) \quad \underline{\mu}_2 = 2[(\underline{g}\underline{\mu})I - M](I-G+\tilde{G})^{-1}\underline{\mu} + (I-G+\tilde{G})[I-A+\tilde{G}-\Delta(\underline{\beta})\tilde{G}]^{-1}\{\underline{\beta}_2^* + 2(\underline{g}\underline{\mu})\underline{\delta} + (\underline{g}\underline{\mu})^2\underline{\beta}_2 + 2[C(1)+(\underline{g}\underline{\mu})B(1)-(\underline{g}\underline{\mu})I](I-G+\tilde{G})^{-1}\underline{\mu}\},$$

and

$$(21)\quad \underline{\nu}_2 = 2[(1-\rho)^{-1}I-N](I-G+\tilde{G})^{-1}\underline{\nu}+2(1-\rho)^{-1}\rho\underline{\nu} + (I-G+\tilde{G})[I-A+\tilde{G}-\Delta(\underline{\beta})\tilde{G}]^{-1}\{(1-\rho)^{-2}\underline{\beta}_2 + 2(1-\rho)^{-1}[B(1)-\rho I](I-G+\tilde{G})^{-1}\underline{\nu}\}.$$

Proof

The proofs of both formulas are similar, so we shall only give the main computational steps of the proof of (20). Several non-obvious steps are involved.

Multiplying the left hand side of (13) on the right by $\underline{e}$, we obtain:

$$(22)\quad \underline{\mu}_2-\sum_{n=1}^{\infty} A_n \sum_{\nu=0}^{n-1} G^{\nu}\underline{\mu}_2 = \underline{\mu}_2-[A-G+\Delta(\underline{\beta})\tilde{G}](I-G+\tilde{G})^{-1}\underline{\mu}_2 = [I-A+\tilde{G}-\Delta(\underline{\beta})\tilde{G}](I-G+\tilde{G})^{-1}\underline{\mu}_2.$$

Multiplying the terms in the right hand side by $\underline{e}$, the first term clearly yields $\underline{\beta}_2^*$. The second term yields

$$(23)\quad 2\sum_{n=1}^{\infty} A_n^{*\prime}(0+)\sum_{\nu=0}^{n-1} G^{\nu}\underline{\mu} = 2[A^{*\prime}(0+)-\sum_{n=0}^{\infty} A_n^{*\prime}(0+)G^n+\sum_{n=1}^{\infty} nA_n^{*\prime}(0+)\tilde{G}](I-G+\tilde{G})^{-1}\underline{\mu} = 2[E(1)-C(1)](I-G+\tilde{G})^{-1}\underline{\mu}-2(\underline{g}\underline{\mu})\underline{\delta}.$$

The third term is more involved. We obtain

$$(24)\quad 2\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2}\sum_{r=0}^{\nu} G^r M G^{\nu-r}\underline{\mu} = 2\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2}\sum_{r=0}^{\nu} G^r M(G^{\nu-r}-G^{\nu+1-r}+\tilde{G})(I-G+\tilde{G})^{-1}\underline{\mu} =$$

$$2\sum_{n=1}^{\infty} A_n \sum_{\nu=0}^{n-1} G^{\nu}M(I-G+\tilde{G})^{-1}\underline{\mu}-2\sum_{n=1}^{\infty} A_n \sum_{\nu=0}^{n-1} G^{\nu}MG^{n-1-\nu}(I-G+\tilde{G})^{-1}\underline{\mu} +2\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2}\sum_{r=0}^{\nu} G^r M\tilde{G}\underline{\mu}.$$

The first two of the latter terms simplify to

$$(25)\quad 2[A-G+\Delta(\underline{\beta})\tilde{G}](I-G+\tilde{G})^{-1}M(I-G+\tilde{G})^{-1}\underline{\mu}$$

$$-2[M-E(1)](I-G+\tilde{G})^{-1}\underline{\mu} = 2E(1)(I-G+\tilde{G})^{-1}\underline{\mu}$$

$$-2[I-A+\tilde{G}-\Delta(\underline{\beta})\tilde{G}](I-G+\tilde{G})^{-1}M(I-G+\tilde{G})^{-1}\underline{\mu}.$$

The last term in (24) yields

$$(26)\quad 2(\underline{g}\underline{\mu})\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2}\sum_{r=0}^{\nu} G^r\underline{\mu} =$$

$$2(\underline{g}\underline{\mu})\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2}[I-G^{\nu+1}+(\nu+1)\tilde{G}](I-G+\tilde{G})^{-1}\underline{\mu},$$

but

$$(27)\quad \sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2}[I-G^{\nu+1}+(\nu+1)\tilde{G}] = \sum_{n=2}^{\infty} nA_n - \sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-1} G^{\nu}$$

$$+\frac{1}{2}\sum_{n=2}^{\infty} n(n-1)A_n\tilde{G} = B(1)-[A-G+\Delta(\underline{\beta})\tilde{G}](I-G+\tilde{G})^{-1}+\frac{1}{2}\Delta(\underline{\beta}_2)\tilde{G},$$

so that the expression in (26) becomes

$$(28)\quad (\underline{g}\underline{\mu})^2\underline{\beta}_2+2(\underline{g}\underline{\mu})[B(1)-I](I-G+\tilde{G})^{-1}\underline{\mu}$$

$$+2(\underline{g}\underline{\mu})[I-A+\tilde{G}-\Delta(\underline{\beta})\tilde{G}](I-G+\tilde{G})^{-2}\underline{\mu}.$$

After collecting terms and multiplying by the inverse of the coefficient matrix in Formula (22), we obtain the explicit formula (20).

<u>Corollary 1</u>

In the M/SM/1 queue with arrival rate λ and service time semi-Markov matrix $\hat{A}(\cdot)$, we have the simplified formulas

$$(29)\quad \rho=\lambda\underline{\pi}\,\underline{\beta}^*, \qquad B(1)=\lambda C(1), \qquad B(2)=\lambda^2 C(2),$$

$$\underline{\beta}=\lambda\underline{\beta}^*, \qquad \underline{\beta}_2=\lambda^2\underline{\beta}_2^*, \qquad \underline{\delta}=\lambda\underline{\beta}_2^*$$

$$\underline{g}\underline{\mu}=\lambda^{-1}\rho(1-\rho)^{-1}, \qquad \underline{\nu}-\lambda\underline{\mu}=\underline{e},$$

$$\underline{\mu}_2=2[\lambda^{-1}\rho(1-\rho)^{-1}I-M](I-G+\tilde{G})^{-1}\underline{\mu}+$$
$$(I-G+\tilde{G})[I-A+\tilde{G}-\lambda\Delta(\underline{\beta}^*)\tilde{G}]^{-1}\{(1-\rho)^{-2}\underline{\beta}_2^*+$$
$$2(1-\rho)^{-1}[C(1)-\lambda^{-1}\rho I](I-G+\tilde{G})^{-1}\underline{\mu}\},$$

$$\underline{\nu}_2=2[(1-\rho)^{-1}I-N](I-G+\tilde{G})^{-1}\underline{\nu}+2(1-\rho)^{-1}\rho\underline{\nu}+$$
$$(I-G+\tilde{G})[I-A+\tilde{G}-\lambda\Delta(\underline{\beta}^*)\tilde{G}]^{-1}\{(1-\rho)^{-2}\lambda^2\underline{\beta}_2^*+$$
$$2(1-\rho)^{-1}[\lambda C(1)-\rho I](I-G+\tilde{G})^{-1}\underline{\nu}\}.$$

Remark

In the scalar case (M/G/1), where $A=G=\tilde{G}=1$, $\lambda\beta^*=\rho$, $\nu=(1-\rho)^{-1}$, $\mu=\lambda^{-1}\rho(1-\rho)^{-1}$, the latter two formulas reduce to the classical formulas.

$$\mu_2=(1-\rho)^{-3}\beta_2^*, \qquad \nu_2=(1-\rho)^{-3}\lambda^2\beta_2^*+2\rho(1-\rho)^{-2}.$$

IV. Numerical Computation

Assuming the required moment matrices, other than E(1) and E(2), defined in (8) are known, the first step is the numerical computation of G. As discussed in [1], we found that successive substitutions in the equation

$$(30)\quad G=(I-A_1)^{-1}\sum_{\substack{\nu=0\\ \nu\neq 1}}^{\infty} A_\nu G^\nu,$$

starting with G=0, converge rapidly except for ρ very close to one.

The evaluation of $\tilde{G}$ and E(1) is routine. It is not necessary to compute the inverse of $(I-G+\tilde{G})$, since only the vectors

$(I-G+\tilde{G})^{-1}\underline{\mu}$ and $(I-G+\tilde{G})^{-1}\underline{\nu}$ are needed and by Formula (10), it is clear that these can be evaluated without inverting the matrix $I-G+\tilde{G}$.

The main computational effort goes into the evaluation of the matrices M and N. As shown in [1], the equations (9) are each equivalent to systems of m^2 linear equations in m^2 unknowns. Except for very small values of m, it is impractical however to compute the required coefficient matrix.

We have successfully computed matrices M and N for m as large as thirty, without expending excessive computer times by straightforward successive substitutions and by using the first of the recurrence relations in (19) to compute the required matrices U_n.

The speed of convergence can be substantially improved by writing the equations (9) in the form

$$(31)\quad X=\left[I-\sum_{n=1}^{\infty} A_n G^{n-1}\right]^{-1}\left[Y+\sum_{n=2}^{\infty} A_n \sum_{\nu=0}^{n-2} G^{\nu} X G^{n-1-\nu}\right],$$

with X=M, Y=E(1) and X=N, Y=G, respectively.

Acknowledgement

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References

[1] Neuts, M. F. (1976)
Moment Formulas for the Markov Renewal Branching Process
Adv. Appl. Prob., 8, 690-711.

[2] Neuts, M. F. (1977)
Some Explicit Formulas for the Steady-state Behavior of the Queue with Semi-Markovian Service Times
Adv. Appl. Prob., 9, 141-157.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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